MODULE- 3

Planar surfaces: {Ref: EM- Bansal, Khurmi, D S Kumar, Benjamin, Bhavikatti

Planar surfaces or plane figure or plane area or lamina have only area but no mass.
eg:- triangle, circle etc

Centroid (G):

The geometric centre of an area of a plane figure is known as centroid. A plane area has only one centroid. Centroid may be inside or outside the body. The moment of area about axis passing through the centroid is zero.

Centre of Gravity (C.G):

Centre of gravity is the point through which the whole weight of the body acts. A body is having only one centre of gravity for all positions of the body. It is the point at which the total mass is assumed to be concentrated.

Centroid applies to plane figures whereas centre of gravity applies to bodies with mass and weight. Centroid and centre of gravity is the same point for a planar surface.

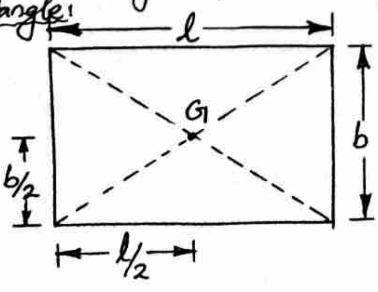
Methods to determine Centroid or C.G:

i) By geometrical considerations
ii) By the method of moments
iii) By the method of integration

i) **By geometrical considerations-**

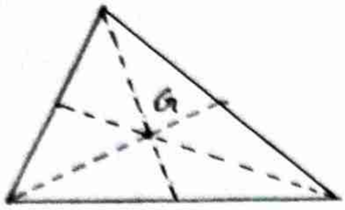
The CG of simple figures may be found out from the geometry of the figure.

1) **Rectangle:**



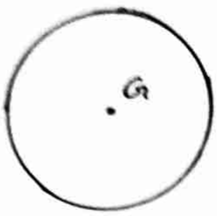
G- point where the diagonals meet.

2) Triangle:



G- point where the three medians meet

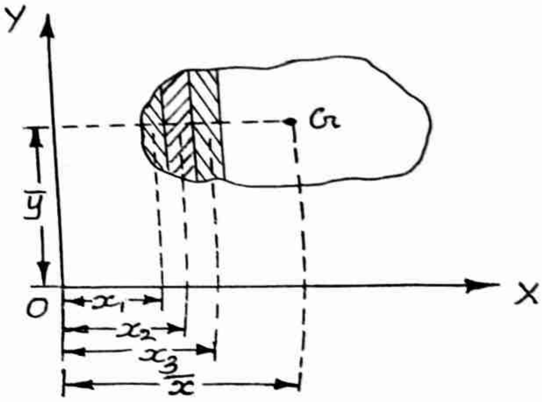
3) Circle:



G- centre of the circle

ii) By the method of moments -

Consider a lamina of area - A. Let this area be divided into a number of small areas - a_1, a_2, a_3 etc. Let $\bar{x}$ and $\bar{y}$ be the co-ordinates of the C.G of the area.



From the principle of moments,

$$\left.\begin{array}{l}\text{Moment of the total} \\ \text{area about OY-axis}\end{array}\right\} = \left\{\begin{array}{l}\text{Sum of} \\ \text{moments of all small areas about OY-axis}\end{array}\right.$$

ie, $A\bar{x} = a_1x_1 + a_2x_3 + a_3x_3 + \ldots\ldots$

where $A = a_1 + a_2 + a_3 + \ldots\ldots$

$$\therefore \bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + \ldots\ldots}{A}$$

ie, $\boxed{\bar{x} = \frac{\Sigma a_i x_i}{\Sigma a_i}}$ |||rly, $\boxed{\bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i}}$

iii) By method of integration -

The co-ordinates of the centroid can be given by;

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int x\,dA}{A} \quad ; \quad \bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int y\,dA}{A}$$

$\int y\,dA$ - First moment of area w.r.t x-axis

$\int x\,dA$ - First moment of area w.r.t y-axis

Axis of Reference:

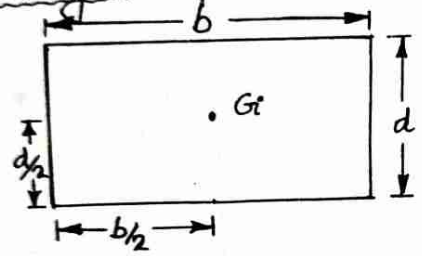
The C.G is calculated with reference to some axis and is known as the axis of reference. The axis of reference, of plane figures is generally taken as the lowest line of the figure for calculating $\bar{y}$ and the left most line of the figure for calculating $\bar{x}$.

C.G of symmetrical sections

If the given section is symmetric about X-X axis or Y-Y axis, the CG will lie on the axis of symmetry.

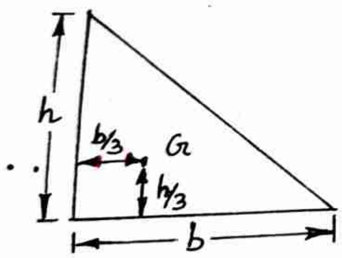
Centroid of common figures:

1) Rectangle:



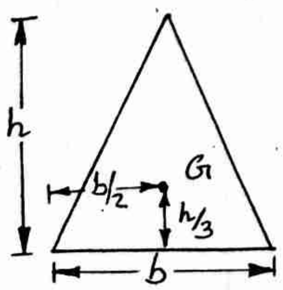
Area = $b \times d$

2) Right angled triangle:



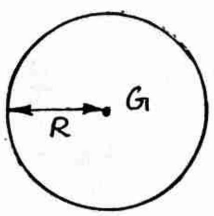
Area = $\frac{1}{2} \times b \times h$

3) Isosceles triangle:



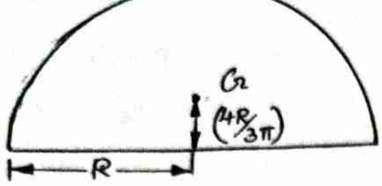
Area = $\frac{1}{2} \times b \times h$

4) Circle:



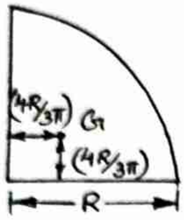
Area = πR^2

5) Semi-circle:



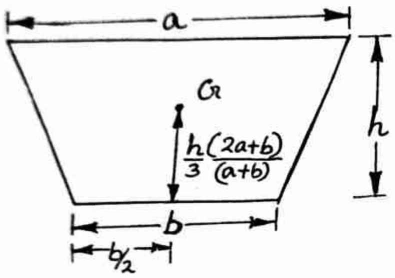
$Area = \frac{\pi R^2}{2}$

6) Quarter circle:



$Area = \frac{\pi R^2}{4}$

7) Trapezium (isosceles):



a – top width

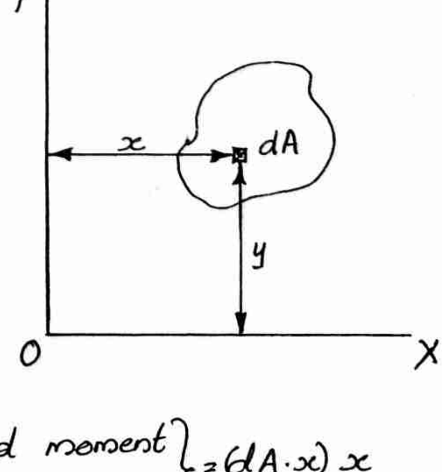
b – bottom width

$Area = \left(\frac{a+b}{2}\right) h$

Area Moment of Inertia or Moment of Inertia –

The second moment of an area about an axis is known as area moment of inertia. It is a geometrical property of an area which reflects how its points are distributed with regard to an arbitrary axis. It is denoted by 'I'.

Consider an elementary area-dA at a distance of 'y' from the x-axis and 'x' from y-axis.



Moment of area or first moment of area-dA about axis-OY $= dA \cdot x$

(First moment of an area is used to determine the CG of an area)

Moment of moment of the area or second moment of the area-dA about the axis-OY $= (dA \cdot x)\, x = dA \cdot x^2$

∴ The second moment of the entire area about the axis - OY $\}= \Sigma dA \cdot x^2$

Illrly, The second moment of the entire area about the axis - OX $\}= \Sigma dA \cdot y^2$

ie, $I_{YY} = \Sigma dA \cdot x^2$ & $I_{XX} = \Sigma dA \cdot y^2$

Hence the magnitude of the moment of inertia is given as the product of the area and the square of the distance of the C.G of the area from the given axis.

Significance of Moment of Inertia -

It is an indication of the resistance of the cross-section against bending or external moment. Higher the moment of inertia, the greater the resistance to bending.

THEOREMS OF MOMENT OF INERTIA

There are 2 theorems of moment of inertia;

i) perpendicular axis theorem

ii) parallel axis theorem

Perpendicular Axis Theorem -

It states that the moment of inertia of an area about an axis perpendicular to its plane at any point is equal to the sum of moment of inertia about any two mutually perpendicular axis through the same point and lying in the plane of the area.

ie, Let I_{ZZ} = MI about Z-axis

I_{XX} = MI about X-axis

I_{YY} = MI about Y-axis

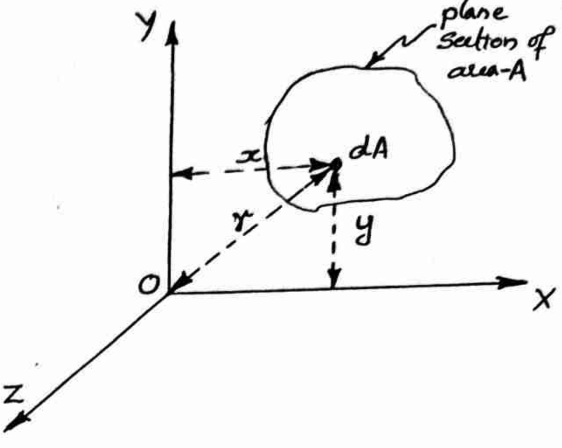
According to perpendicular axis theorem,

$$\boxed{I_{ZZ} = I_{XX} + I_{YY}}$$

I_{ZZ} is also known as polar moment of inertia.

Proof:-

Consider a plane section of area A lying in the x
Let OX and OY be the 2 mutually perpendicular axes
and OZ be the axis ⊥r to the plane of the section.



Consider a small area-dA.

Let x- distance of dA from the axis OY.

y- distance of dA from the axis OX

r - distance of dA from the axis OZ

Then $r^2 = x^2 + y^2$

Moment of inertia of dA about X-axis $= dA \times y^2$

Moment of inertia of total area-A about X-axis, $I_{XX} = \Sigma dA y^2$

Illy, Moment of inertia of total area-A about Y-axis, $I_{YY} = \Sigma dA \cdot x^2$

Moment of inertia of total area-A about Z-axis, $I_{ZZ} = \Sigma dA \cdot r^2$

$$\text{ie, } I_{ZZ} = \Sigma dA \cdot r^2 = \Sigma dA\,(x^2+y^2)$$

$$= \Sigma dA x^2 + \Sigma dA y^2$$

$$\text{ie, } \underline{I_{ZZ} = I_{YY} + I_{XX}}$$

Parallel Axis Theorem-

It states that the moment of inertia of an area about any axis in the plane of that area is equal to the sum of moment of inertia about a parallel axis through its centroid and product of area and square of the distance between two parallel axes

According to parallel axis theorem,

$$\boxed{I_{AB} = I_G + Ah^2}$$

where I_{AB} - MI of the given area about axis-AB

I_G - MI of the given area about centroidal axis parallel to AB

A- Area of the section

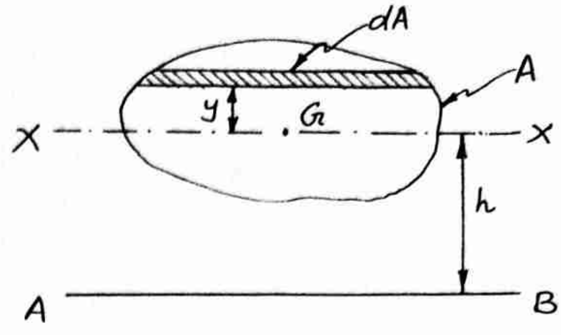
h- Distance between CG and axis AB

Proof:-

Consider a plane area-A.
Let XX-axis passing through the C.G of the plane.

AB- axis parallel to XX axis

h - Distance b/w AB and XX-axis



Consider a strip of area-dA at a distance-y from XX-axis.

Moment of inertia of area-dA about XX-axis = $dA \cdot y^2$

Moment of inertia of total area about XX-axis, I_{XX} or $I_G = \Sigma dA \cdot y^2$

Moment of inertia of area-dA about AB = $dA(h+y)^2$

$= dA(h^2+y^2+2hy)$

Moment of inertia of total area about- AB,

$$I_{AB} = \Sigma dA(h^2+y^2+2hy)$$

$$= \Sigma dA h^2 + \Sigma dA y^2 + \Sigma dA \cdot 2hy$$

Since 'h' is a constant,

$$I_{AB} = h^2 \Sigma dA + \Sigma dA y^2 + 2h \cdot \Sigma dA y$$

But $\Sigma dA = A$ & $\Sigma dA y^2 = I_G$

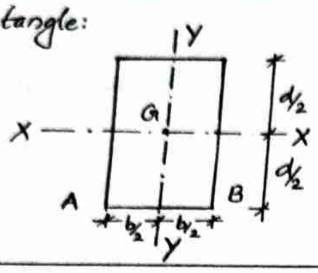
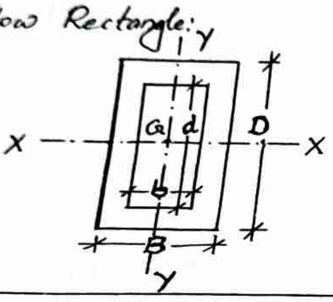
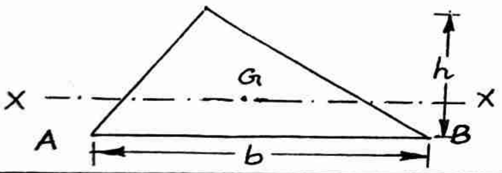
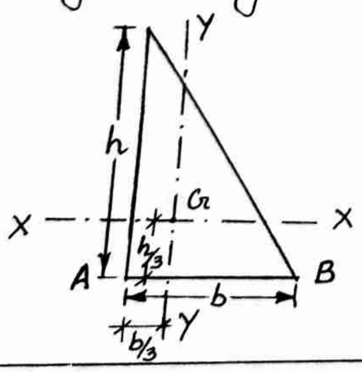
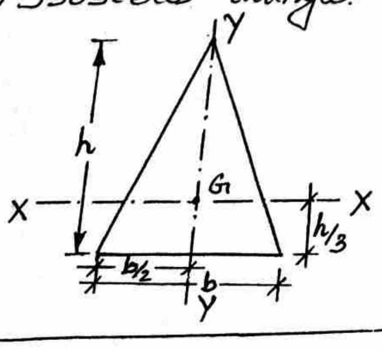
ie, $I_{AB} = Ah^2 + I_G + 2h\, \Sigma dA y$

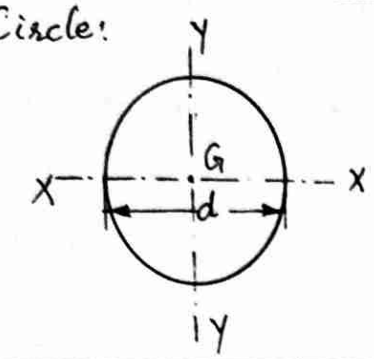
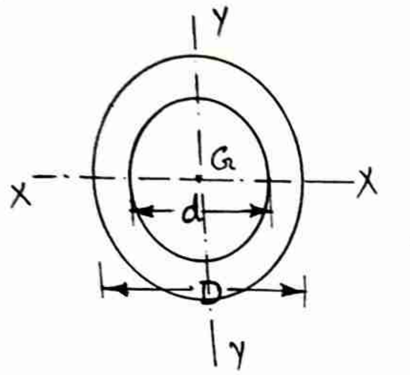
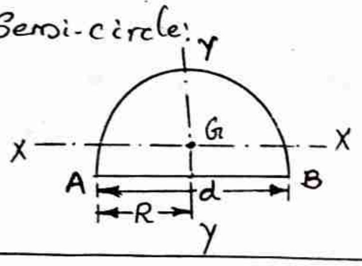
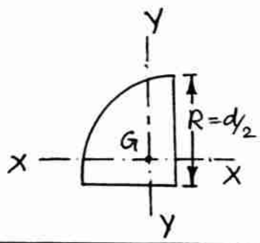
But '$dA \cdot y$' represents the moment of strip about XX-axis and hence $\Sigma dA y$ represents the moment of the total area about XX-axis. But moment of the total area about XX-axis is equal to the product of total area and the distance of the C.G of the total area from XX-axis. As the distance of the C.G of the total area from XX-axis is zero, hence $\Sigma dA \cdot y = 0$

$\therefore\ I_{AB} = Ah^2 + I_G + 0$

ie, $\underline{I_{AB} = I_G + Ah^2}$

Moment of Inertia of Standard Sections {Ref: EM-

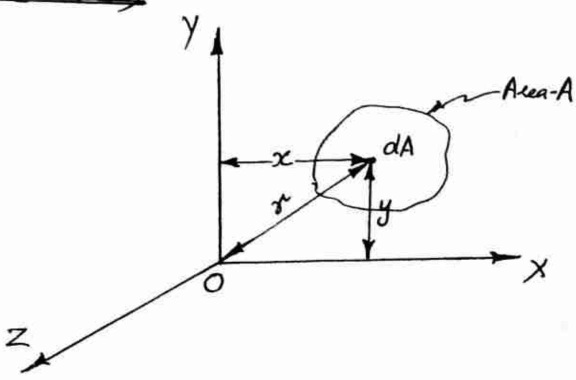
SHAPE	MOMENT OF INERTIA
1) Rectangle: 	$I_{XX} = \frac{bd^3}{12}$ $I_{YY} = \frac{db^3}{12}$ $I_{AB} = \frac{bd^3}{3}$
2) Hollow Rectangle: 	$I_{XX} = \frac{BD^3 - bd^3}{12}$
3) Triangle: Scalene 	$I_{XX} = \frac{bh^3}{36}$ $I_{AB} = \frac{bh^3}{12}$
4) Right triangle: 	$I_{XX} = \frac{bh^3}{36}$ $I_{YY} = \frac{hb^3}{36}$ $I_{AB} = \frac{bh^3}{12}$
5) Isosceles triangle: 	$I_{XX} = \frac{bh^3}{36}$ $I_{YY} = \frac{hb^3}{48}$ $I_{AB} = \frac{bh^3}{12}$

SHAPE	MOMENT OF INERTIA
6) Circle: 	$I_{XX} = I_{YY} = \frac{\pi d^4}{64}$
7) Hollow Circle: 	$I_{XX} = I_{YY} = \frac{\pi}{64}(D^4 - d^4)$
8) Semi-circle: 	$I_{XX} = 0.11R^4$ $I_{YY} = \frac{\pi R^4}{8} = \frac{\pi d^4}{128}$ $I_{AB} = \frac{\pi R^4}{8} = \frac{\pi d^4}{128}$
9) Quarter circle: 	$I_{XX} = 0.055R^4$ $= 0.00343d^4$ $I_{YY} = 0.055R^4$ $I_{AB} = \frac{\pi R^4}{16} = \frac{\pi d^4}{256}$

Polar Moment of Inertia (J or I_{ZZ})-

Polar moment of inertia of an area w.r.t an axis ⊥r to the plane is equal to the sum of the moments of inertia about two mutually ⊥r axes in the plane of the area.

ie, J or $I_{ZZ} = I_{XX} + I_{YY}$



where I_{xx} - M.I about X-axis

I_{yy} - M.I about Y-axis

I_{zz} - M.I about Z-axis

Polar moment of inertia is also known as the second moment of area about Z-axis.

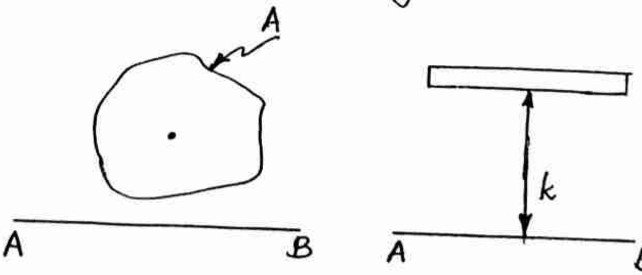
ie, J or $I_{zz} = \Sigma dA \cdot r^2$

The product of the area and the square of the distance of the centre of gravity of the area from an axis ⊥r to the plane of the area is polar moment of inertia.

Radius of gyration: {EM: Bansal, P.K. Sasidharan

Consider an area-A which has a moment of inertia-I with respect to a reference axis AB. Let us assume that this area is compressed to a thin strip parallel to axis-AB. For this strip to have the same moment of inertia I, with respect to the same reference axis-AB, the strip should be placed at a distance-k from the axis AB such that $I = Ak^2$. This distance 'k' is known as the radius of gyration of the area with respect to the given axis-AB.

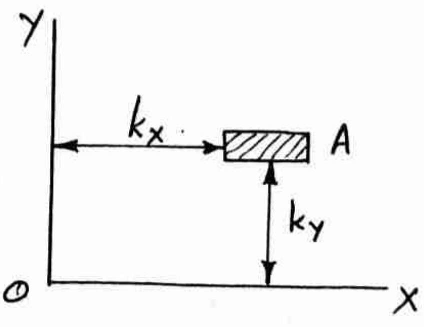
Radius of gyration of a given lamina about an axis is a distance such that its square multiplied by the area gives the moment of inertia of the area about the given axis.



ie, $I = Ak^2$; where k - radius of gyration about the given axis.

$$\Rightarrow k = \sqrt{\frac{I}{A}}$$

Also, $k_x = \sqrt{\frac{I_{yy}}{A}}$; $k_y = \sqrt{\frac{I_{xx}}{A}}$



Radius of gyration does not

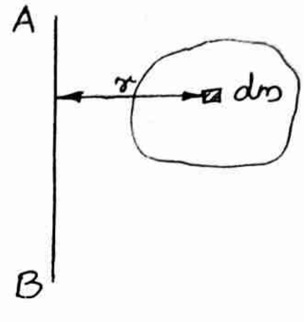
identify a physical point on the area-A. It is defined as the distance from the axis to a point where the concentrated area of the same size could be placed to have same moment of inertia with respect to the given axis.

Mass Moment of Inertia {Ref:- EM · Bhavikatti, Rajasekaran, D.S. Kumar (7)
(I or I_m)

Moment of inertia of a physical body is termed as mass moment of inertia.

The mass moment of inertia of the body shown in figure about axis-AB is given by,

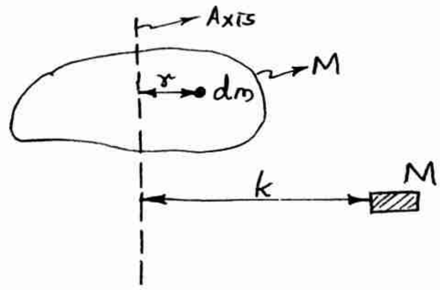
$$I_{AB} = \Sigma dm\, r^2 = \int r^2 dm \; ; \underline{\text{unit}-kgm^2}$$



where 'r' is the distance of the elementary mass-dm from AB. Hence, mass moment of inertia is the product of mass and the square of the distance of the CG of the mass from an axis.

The term mass moment of inertia has no physical meaning. It is only a mathematical term, which is useful in studying rotation of rigid bodies. It is represented by I_m or I.

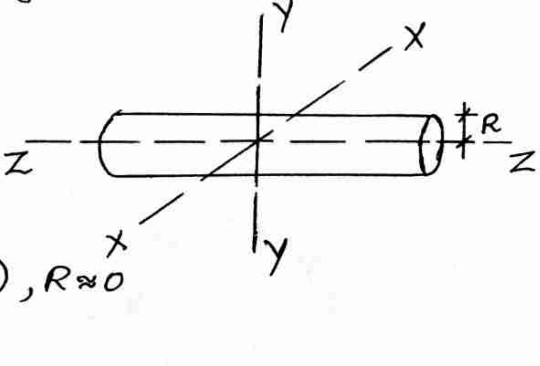
The radius of gyration of the bodies w.r.t the axis is given by, $k = \sqrt{I/m}$; where I - mass moment of inertia
M - mass of the body



Radius of gyration is the distance at which the entire mass-M of the body is assumed to be concentrated from the axis such that the moment of inertia of the actual body and the concentrated mass is same.

Mass moment of inertia of a cylinder:

a) Solid cylinder:



$$I_{ZZ} = \frac{MR^2}{2}$$

$$I_{XX} = I_{YY} = \frac{M}{12}(3R^2 + L^2)$$

Note: For a slender rod (thin cylinder), $R \approx 0$

$$\therefore I_{XX} = I_{YY} = \frac{ML^2}{12}$$

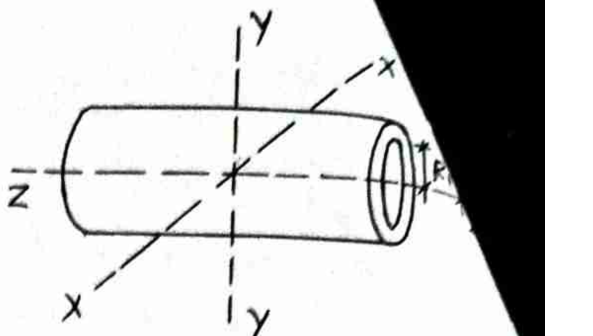
For a thin disc, $L = 0$

$$\therefore I_{XX} = I_{YY} = \frac{MR^2}{4}$$

b) Hollow Cylinder:

$$I_{xx} = I_{yy} = \frac{M}{12}\left[3(R_1^2+R_2^2)+L^2\right]$$

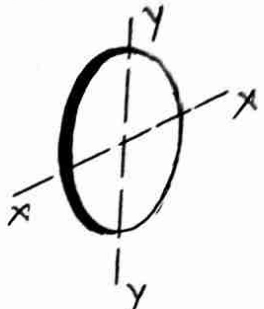
$$I_{zz} = \frac{M}{2}(R_1^2+R_2^2)$$



Mass moment of inertia of a thin circular disc:

$$I_{xx} = I_{yy} = \frac{MR^2}{4}$$

$$I_{zz} = \frac{MR^2}{2}$$



Mass moment of inertia of composite bodies: {Ref: EM-Benjamin

The composite body can be divided into a set of simple bodies. Moment of inertia of each body about its centroidal axis can be calculated using the parallel axis theorem, moment of inertia of each body can be calculated about the required axis. Summing up of M.I of each simple body gives the moment of inertia of the composite body.

Parallel axis theorem formula:

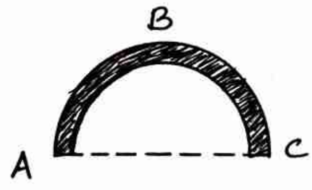
$$I_A = I_G + Md^2$$

THEOREMS OF PAPPUS AND GULDINUS-

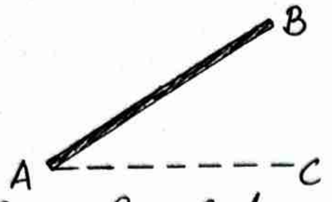
The theorems offer a simple way for computing the area of surface of revolution and the volume of bodies of revolution.

A surface of revolution is a surface which may be generated by rotating a plane curve about a fixed axis.

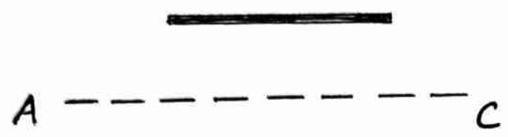
eg:- i) Surface of sphere- rotating a semi-circular arc ABC about axis-AC.



ii) Surface of cone - rotating inclined line - AB about axis - AC.

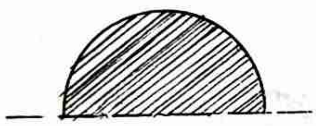


iii) Surface of cylinder - rotating a horizontal line about axis - AC.



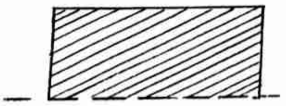
A body of revolution is the body which is generated by rotating a plane area about a fixed axis.

eg:- i) Solid sphere - rotating a semi-circular area about the axis.



ii) Cone - rotating a triangular area about the axis.

iii) Cylinder - rotating a rectangular area about the axis.

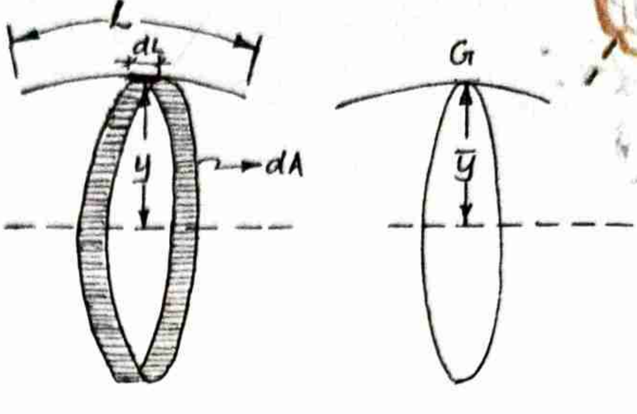


Theorem - 1

The area of surface generated by revolving a plane curve about a non-intersecting axis in the plane of the curve is equal to the product of length of curve and the distance travelled by the centroid of the curve while the surface is being generated.

Proof:-

Consider an element of length - dL of the curve of length L which is revolved about the x-axis. The area generated by the element is equal to $2\pi y \cdot dL$, where 'y'- distance of element from x-axis.



Total area generated by the curve,

$$A = \int 2\pi y \, dL = 2\pi \int y \, dL$$

$$\underline{A = (2\pi \bar{y}) L}$$

(Since $\bar{y} L = \int y \, dL$, eqn for centroid of curve)

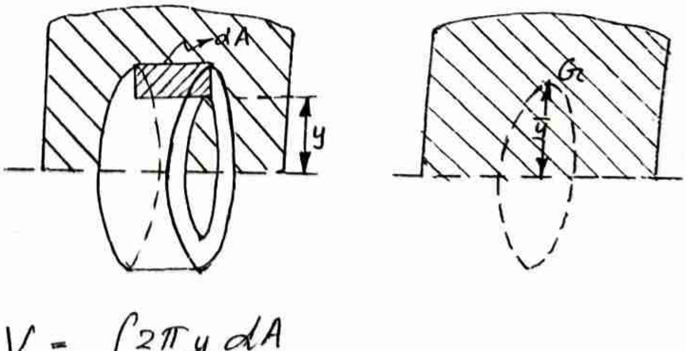
$2\pi\bar{y}$ - distance travelled by the centroid of the curve of length-L.

Theorem-2

The volume of a body generated by revolving a plane area about a non intersecting axis in the plane of the area is equal to the product of area and the distance travelled by the centroid of the plane area while the body is being generated.

Proof:

Consider an element-dA of the area - A which is revolved about the x-axis.



Volume generated by the elemental area = $2\pi y \, dA$

Total volume generated, $V = \int 2\pi y \, dA$

$$= 2\pi \int y \, dA$$

$$\underline{V = (2\pi \bar{y}) A}$$

(Since $\int y \, dA = \bar{y} A$, equation for centroid of area)

$2\pi\bar{y}$ - distance travelled by the centroid of area - A.